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Five-dimensional gauge theories and quantum mechanical matrix models

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Abstract: We show how the Dijkgraaf-Vafa matrix model proposal can be extended to describe five-dimensional gauge theories compactified on a circle to four dimensions. This involves solving a certain quantum mechanical matrix model. We do this for the lift of the $\mathcal{N} = 1^*$ theory to five dimensions. We show that the resulting expression for the superpotential in the confining vacuum is identical with the elliptic superpotential approach based on Nekrasov's five-dimensional generalization of Seiberg-Witten theory involving the relativistic elliptic Calogero-Moser, or Ruijsenaars-Schneider, integrable system.

Keywords: Supersymmetry and Duality, Supersymmetric Effective Theories, Matrix Models, Integrable Field Theories.

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1. Introduction

The Dijkgraaf-Vafa approach for calculating the superpotential of a four-dimensional supersymmetric gauge theory via a matrix model [1, 2, 3] is by now well established. The matrix model seems capture all the holomorphic information of an $\mathcal{N} = 1$ theory in a lower dimensional — in this case a zero dimensional — system. Recently Dijkgraaf and Vafa [4] have proposed a much more general form of their approach which would apply, for instance, to higher dimensional theories compactified to four dimensions. This far-reaching proposal is very striking and without doubt deserves detailed study and independent testing.

From the original proposal involving matrix integrals, it has turned out that relevant perturbations of the $\mathcal{N} = 4$ theory, the so-called $\mathcal{N} = 1^*$ theory, [3, 5, 6], and its Leigh-Strassler deformations [7, 8], can be solved since the associated matrix models were studied in other contexts [9, 10]. For the $\mathcal{N} = 1^*$ theory the matrix model results agree with the elliptic superpotential approach which is an alternative approach for calculating the holomorphic condensates [11, 12]. In order to investigate the recent higher-dimensional proposal, it is natural to consider relevant deformations of the $\mathcal{N} = 4^*$ theory lifted to five dimensions where the additional dimension is compact. The recent work of Dijkgraaf and Vafa would have us consider a quantum mechanical matrix model rather than a matrix integral. We shall solve this model in section 2 and hence compute the superpotential in the confining vacuum of the theory. The results in the other massive vacua follow by a straightforward generalization of [6].

Fortunately we have an independent and highly non-trivial check on the result based on the elliptic superpotential approach adopted in [11, 12]. The generalization to the five dimensional theory has not been described previously but it not difficult to establish the form of the exact elliptic superpotential in this case, generalizing the one for the $\mathcal{N} = 1^*$ theory in [11]. The point is that the Seiberg-Witten theory of the compactified five-dimensional theory was considered by Nekrasov [13]. What happens is that the Calogero-Moser integrable system which underlies the four dimensional theory is replaced by its relativistic version: the so-called Ruijsenaars-Schneider system [14]. The exact elliptic superpotential

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for the simplest mass deformation of the $\mathcal{N}=2$ theory is identified with the basic hamiltonian of this integrable system. Hence we are able to independently calculate, for instance, the gluino condensate in the confining vacuum of the compactified five dimensional theory and the result is entirely consistent with the expression calculated via matrix quantum mechanics.

As a side remark, we shall find that the quantum-mechanical matrix model can be reduced to a matrix integral and this integral is very similar to the one that described the relevant deformations of the Leigh-Strassler theory in [7]. It is rather natural, therefore, that the elliptic superpotential side of the story should involve the same integrable system [8]; namely, that of Ruijsenaars and Schneider. Surely this is a strong hint of some deeper connection between the integrable system and the matrix model.

2. The quantum-mechanical matrix model

We begin by formulating the superpotential of the compactified five-dimensional theory as a quantum mechanical matrix model following Dijkgraaf and Vafa [4]. From the point-of-view of the four-dimensional theory there are 3 adjoint chiral fields Φ_i, $i=1,2,3$. In the five-dimensional theory, the imaginary component of, say, Φ_3 is re-interpreted as the component of the gauge field along the extra compact dimension, while the real part of Φ_3 is the real scalar of the five-dimensional theory. We will take the compact dimension to have length β. The other fields Φ_1 and Φ_2 form an adjoint-valued hypermultiplet. The superpotential of the effective four-dimensional theory is determined by the quantum mechanical system involving the fields $\Phi_i(t)$. The partition function of the quantum mechanical system involves the functional integral:

$$Z=\int\prod_{i=1}^{3}[d\Phi_i]\,\exp\left(-(\beta g_s)^{-1}\int dt\,W[\Phi_i]\right). \tag{2.1}$$

The action of the matrix model is a generalization of the one that describes the $\mathcal{N}=1^*$ deformation of the four dimensional theory:

$$W[\Phi_i]=\mathrm{Tr}\left(i\Phi_1 D\Phi_2+m\Phi_1\Phi_2+\mu\cosh(\beta\Phi_3)\right) \tag{2.2}$$

where the covariant derivative is $D\Phi_2=\partial_t\Phi_2+[\Phi_3,\Phi_2]$. In fact we have been a bit implicit in writing down (2.1) because we have not specified the measure. Part of the Dijkgraaf-Vafa prescription involves interpreting the matrix integral in a holomorphic way. So the complex fields $\Phi_1(t)$ and $\Phi_2(t)$ are subject to a particular reality condition, namely $\Phi_1^\dagger=\Phi_2$, or equivalently $\Phi_1+\Phi_2$ and $i(\Phi_1-\Phi_2)$ are hermitean, from the point-of-view of the functional integral. In particular the measure for the latter combination of fields is the appropriate measure for hermitean fields. The field $\Phi_3(t)$ is treated in a somewhat different manner as described in [4]. First of all, local gauge transformations can be used to gauge away the non-constant part of the component of the gauge field along the compact direction. This leaves large gauge transformations which shift the eigenvalues of the gauge field by integer multiples of $2\pi/\beta$. This means that the natural variable is not the gauge field, but rather its

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holonomy around the circle $\exp i\beta A_t$. In the holomorphic point-of-view the quantity iA_t is then naturally complexified to Φ_3 by including the real scalar and then the prescription of Dijkgraaf and Vafa [4] is to interpret the integral as being over the t-independent quantity

$$U = \exp \beta\Phi_3 \tag{2.3}$$

thought of as a unitary matrix. Notice that the final term in (2.2) is the natural generalization of the $\mathrm{Tr}\,\Phi_3^2$ in the four-dimensional theory and incorporates the necessary periodicity $\Phi_3 \to \Phi_3 + 2\pi i/\beta$ and will allow us to make contact with the generalization of the condensate u_2 described in [13].

Now following the logic of [3, 5], we integrate out the fields $\Phi_1(t)$ and $\Phi_2(t)$ since they appear gaussian in (2.1). The complication is that we must now integrate out all the fourier modes of these fields. We end up with a pure unitary (zero-dimensional) matrix integral

$$Z = \int dU \, \frac{\exp\big(-g_s^{-1}\mu\,\mathrm{Tr}\cosh(\beta\Phi_3)\big)}{\det\sinh\frac{1}{2}\big(im + \Phi_3\otimes\mathbf{1} - \mathbf{1}\otimes\Phi_3\big)}\,, \tag{2.4}$$

where we used the identity

$$\det(\partial_t + \varphi) = \det\,\sinh\frac{1}{2}(\beta\varphi)\,\Big/\left(\frac{1}{2}\beta\right). \tag{2.5}$$

Now we are ready to perform the large-N limit saddle-point evaluation of the remaining unitary integral around the critical point appropriate to the confining vacuum. As usual in a unitarity matrix integral we can diagonalize U and work in terms of its eigenvalues

$$U \sim \big(e^{\phi_1}, \ldots, e^{\phi_N}\big) \tag{2.6}$$

at the expense of introducing the unitary matrix version of the Vandermonde determinant:

$$Z = \int \prod_i d\phi_i \prod_{i\neq j} \frac{\sinh\frac{1}{2}(\phi_i - \phi_j)}{\sinh\frac{1}{2}(\phi_i - \phi_j + i\beta m)} \, \exp\left(-g_s^{-1}\mu\sum_i \cosh\phi_i\right). \tag{2.7}$$

Apart from the potential, this is precisely the matrix integral that appears in the solution of the six vertex model on a random lattice [10] and has already been employed to describe relevant perturbations of Leigh-Strassler deformations of $\mathcal{N}=4$ theories. However, it is almost identical to the matrix model that was considered in [15] which arose from taking a dimensional reduction of $\mathcal{N}=1$ supersymmetric Yang-Mills in four dimensions to one dimension. The only difference is that the periodicity in the eigenvalues in that reference is along the real axis, however, this is only a superfical difference because our integral is interpreted in a holomorphic way. In addition, our model is also identical to the different relevant deformation of the Leigh-Strassler theory considered in [16]. In order to solve the model, we will largely follow the approach of [7] (or [16]) which is tailored towards the Dijkgraaf-Vafa application.

In the large-N limit, the eigenvalues ϕ_i form a continuum and condense onto cuts in the complex z-plane (actually the cylinder due to the identification $z \sim z + 2\pi i$). One can think of these cuts as arising from a quantum smearing-out of the classical eigenvalues.

For the confining vacuum all the classical eigenvalues are degenerate $\phi_i = 0$ and so we expect a solution in the matrix model involving a single cut which we take to extend from $-a$ to a. Notice that at this point we diverge from the Leigh-Strassler case where the cut is not symmetrical about $z = 0$ [7]. The extent of the cut and the matrix model density of eigenvalues $\rho(z)$ will be determined self-consistently from the saddle-point equation in terms of the 't Hooft coupling of the matrix model $S = g_s N$. The saddle-point equation is most conveniently formulated after defining the resolvent function

$$\omega(z) = \frac{1}{2}\int_{-a}^{a} d\phi \, \frac{\rho(z)}{\tanh\frac{z-\phi}{2}}\,, \qquad \int_{-a}^{a} \rho(\phi)\, d\phi = 1\,. \tag{2.8}$$

This function is analytic in z and its only singularity is along a branch cut extending between $[-a, a]$. The matrix model spectral density $\rho(\phi)$ is equal to the discontinuity across the cut

$$\omega(\phi + i\epsilon) - \omega(\phi - i\epsilon) = -2\pi i \rho(\phi)\,, \qquad \phi \in [-a, a]\,. \tag{2.9}$$

In this, and following equations, ϵ is an infinitesimal. The saddle-point equation expresses the condition of zero force on a test eigenvalue in the presence of the large-N distribution of eigenvalues along the cut:

$$\frac{\mu \sinh\phi}{S} = \omega(\phi + i\epsilon) + \omega(\phi - i\epsilon) - \omega(\phi + i\beta m) - \omega(\phi - i\beta m)\,, \qquad \phi \in [-a, a]\,. \tag{2.10}$$

This equation can be re-written in terms of the useful function

$$G(z) = \frac{\mu}{2\sin(\frac{\beta m}{2})}\cosh z + iS\left(\omega\left(z + \frac{i\beta m}{2}\right) - \omega\left(z - \frac{i\beta m}{2}\right)\right). \tag{2.11}$$

From this definition, one may easily deduce that $G(z)$ is an analytic function on the cylinder $-\pi \le z \le \pi$, where $\mathrm{Im}\, z = \pi$ and $\mathrm{Im}\, z = -\pi$ are identified, with two cuts $[-a + \frac{i\beta m}{2}, a + \frac{i\beta m}{2}]$ and $[-a - \frac{i\beta m}{2}, a - \frac{i\beta m}{2}]$. This is illustrated in figure 1. Note that everything is periodic in $\beta m \to \beta m + 2\pi$ so we can always choose $-\pi < \beta m < \pi$.

In terms of $G(z)$, the matrix model saddle-point equation (2.10) is

$$G\left(\phi + \frac{i\beta m}{2} \pm i\epsilon\right) = G\left(\phi - \frac{i\beta m}{2} \mp i\epsilon\right)\,, \qquad \phi \in [-a, a]\,. \tag{2.12}$$

This equation can be viewed as a condition which glues the top (bottom) of the upper cut to the bottom (top) of the lower cut thereby defining a torus with two marked points corresponding to the infinities $\mathrm{Re}\, z \to \pm\infty$. The function $G(z)$ is then uniquely specified by gluing condition eq. (2.12) and asymptotic behaviour at large $\pm\mathrm{Re}\, z$

$$\lim_{\mathrm{Re}\, z \to \pm\infty} G(z) \to \frac{\mu}{4\sin(\frac{\beta m}{2})} e^{\pm z} + \mathcal{O}(e^{\mp z})\,, \tag{2.13}$$

which is a consequence of eq. (2.11).

The auxiliary torus $E_{\tilde\tau}$ is specified by a complex structure $\tilde\tau$ which can be thought of as a function of the parameter a specifying the length of the cut. We can uniformize the torus by establishing a map to the complex u-plane (u being an auxiliary variable)

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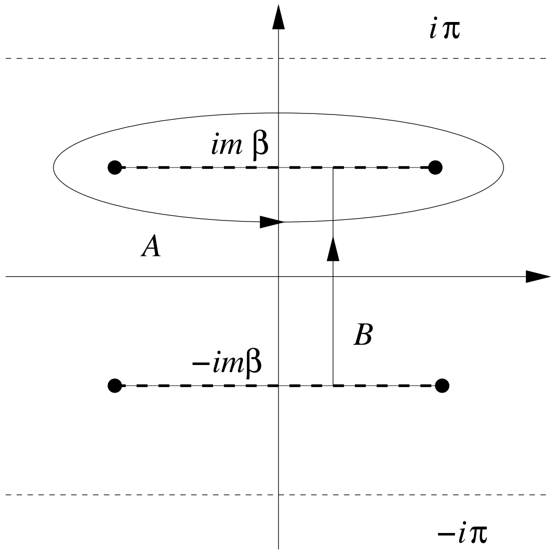

Figure 1: The region over which the function $G(z)$ is defined with its two cuts. The lines $\mathrm{Im}\, z = \pm\pi$ are identified.

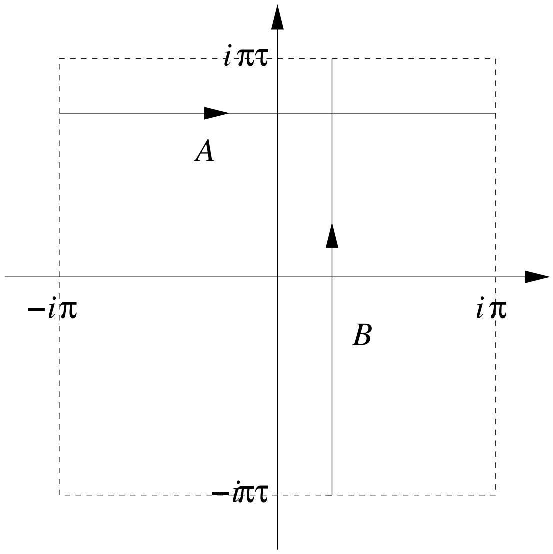

Figure 2: Uniformizing map from the z-plane to the torus $E_{\tilde{\tau}}$.

quotiented by a lattice with periods $2\pi i$ and $2\pi i\tilde{\tau}$. The torus is shown in figure 2, where for illustrative purposes $\tilde{\tau}$ has been taken to be purely imaginary. As shown in figures 1 and 2, the contour A enclosing the cut $[-a + \frac{i\beta m}{2}, a + \frac{i\beta m}{2}]$ anti-clockwise maps to one of the cycles of the torus while the contour B joining the two cuts maps to the conjugate cycle.

The map $z(u)$ from the u-plane to the z-plane is specified by the requirements that going around the contour A returns z to its original value, while traversing the contour B causes z to jump by an amount $i\beta m$, which is the distance between the two cuts in the z-plane. Both these operations leave G unchanged implying that it is an elliptic function on the u-plane. Thus

$$A: \qquad z(u + 2\pi i) = z(u)\,; \qquad G(z(u + 2\pi i)) = G(z(u))\,, \tag{2.14a}$$

$$B: \qquad z(u + 2\pi i\tilde{\tau}) = z(u) + i\beta m\,; \qquad G(z(u + 2\pi i\tilde{\tau})) = G(z(u))\,. \tag{2.14b}$$

This determines the following unique map $z(u)$ from the u-plane to the z-plane:

$$\exp z(u) = \frac{\theta_1(\frac{u}{2i} - \frac{\beta m}{4}|\tilde{\tau})}{\theta_1(\frac{u}{2i} + \frac{\beta m}{4}|\tilde{\tau})}\,. \tag{2.15}$$

The only singularities of $G(z(u))$ are simple poles at $u = \pm\frac{i\beta m}{2}$ corresponding to large $\pm\mathrm{Re}\, z$ in order to incorporate the asymptotic behaviour (2.13). The fact that $G(z(u))$ is elliptic, along with the singularity structure, specifies it uniquely:

$$G(z(u)) = \frac{i}{2\sin\frac{\beta m}{2}} \cdot \frac{\theta_1(\frac{\beta m}{2}|\tilde{\tau})}{\theta_1'(0|\tilde{\tau})}\left(\zeta\left(u - \frac{i\beta}{2}\right) - \zeta\left(u + \frac{i\beta}{2}\right) + 2\zeta(i\beta) - \frac{\beta}{\pi}\zeta(\pi i)\right). \tag{2.16}$$

In (2.15) and (2.16), the quasi-elliptic function $\zeta(u)$ is defined on the torus E_τ and $\theta_1(x|\tau)$ is a Jacobi theta function (see [17] for definitions and appendix A for some useful identities).

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As expected from (2.14b), $G(z(u))$ is an elliptic function of u with two simple poles in the u-plane; $z(u)$ on the other hand is only quasi-elliptic. Having determined $G(z)$ in the elliptic parameterization we can now implement the Dijkgraaf-Vafa proposal in order to compute the superpotential in the confining vacuum.

According the Dijkgraaf-Vafa proposal, the gluino condensate of the gauge theory gets identified with the 't Hooft coupling S of the matrix model. From eqs. (2.8) and (2.11), the integral of $G(z)$ around the contour A is equal to $-2\pi g_s N = -2\pi S$. Under the map to the torus this becomes an integral around the A-cycle:

$$2\pi i S = -i \int_A G(z(u)) \frac{dz(u)}{du} du\,. \tag{2.17}$$

The second ingredient required to determine the QFT superpotential is the variation of the genus zero free energy $\mathcal{F}_0$ of the matrix model in transporting a test eigenvalue from infinity to the endpoint of the cut. This is obtained by integrating the force on a test eigenvalue, which can be expressed in terms of the function $G(z)$ as

$$-i\left(G\left(z+\frac{i\beta m}{2}\right) - G\left(z-\frac{i\beta m}{2}\right)\right), \tag{2.18}$$

from infinity to the original cut $[-a, a]$. This can be written as an integral over $G(z)$ alone along a contour starting at the lower cut, going off to infinity and then back to the upper cut. This can be deformed into the contour B running from the lower cut to the upper cut as in figure 1. Under the map to the torus this becomes an integral over the B-cycle:

$$\frac{\partial \mathcal{F}_0}{\partial S} = -i \int_B G(z(u)) \frac{dz(u)}{du} du\,. \tag{2.19}$$

The effective superpotential in the confining vacuum is obtained by extremizing the following expression with respect to S:

$$W_{\text{eff}} = N \frac{\partial \mathcal{F}_0}{\partial S} - 2\pi i \tau S\,, \tag{2.20}$$

where τ is the bare coupling of the theory not to be confused with the complex structure of $E_{\tilde{\tau}}$; we shall shortly relate the two.

Both integrals (2.17) and (2.19) are evaluated using standard elliptic function identities:

$$2\pi i S = \frac{dh(\tilde{\tau})}{d\tilde{\tau}}\,, \qquad \frac{\partial \mathcal{F}_0}{\partial S} = \tilde{\tau}\frac{dh(\tilde{\tau})}{d\tilde{\tau}} - h(\tilde{\tau})\,, \tag{2.21}$$

where

$$h(\tilde{\tau}) = \frac{\mu}{\sin\frac{\beta m}{2}} \cdot \frac{\theta_1(\frac{\beta m}{2}|\tilde{\tau})}{\theta_1'(0|\tilde{\tau})}\,. \tag{2.22}$$

It follows that

$$\frac{\partial W_{\text{eff}}}{\partial S} = 0 \quad \Longrightarrow \quad \tilde{\tau} = \frac{\tau}{N} \tag{2.23}$$

so that

$$W_{\text{eff}} = -N h\left(\frac{\tau}{N}\right) = -\frac{N\mu}{\sin\frac{\beta m}{2}} \cdot \frac{\theta_1(\frac{\beta m}{2}|\frac{\tau}{N})}{\theta_1'(0|\frac{\tau}{N})}\,. \tag{2.24}$$

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As a test of our expression for the superpotential we can consider the limit $\beta \to 0$ and relate it to the known result [3, 5, 11]. In this limit, from (2.24) we find

$$W_{\text{eff}} \to -N\mu - \frac{N\mu m^2}{24} E_2\left(\frac{\tau}{N}\right)\beta^2 + \cdots , \tag{2.25}$$

where $E_2(\tau)$ is the second Eisenstein series. This result agrees exactly with the expected result [3, 5, 11] since in this limit the potential behaves as $V(\Phi) \to \mu(1 + \frac{1}{2}\beta^2\Phi^2 + \cdots)$.

It is possible to use the quantum mechanical matrix model to calculate the superpotential in all the massive vacua of the theory. The idea is to consider special multi-cut saddle-point solutions as described in [6]. We do not include the details here, but they can straightforwardly be deduced from the aforementioned reference. The final result for the p^{th} massive vacuum, where the confining corresponds to $p = 1$ and the Higgs to $p = N$, is

$$W_{\text{eff}} = -\frac{N\mu}{\sin\frac{\beta m}{2}} \cdot \frac{\theta_1(\frac{p\beta m}{2}|\frac{p^2\tau}{N})}{\theta_1'(0|\frac{p^2\tau}{N})} . \tag{2.26}$$

Recall that the result for the superpotential of the matrix model calculation in the four-dimensional $\mathcal{N} = 1^*$ case does not have exact modular symmetry. For instance, the S transformation $\tau \to -1/\tau$ only relates the superpotential in the Higgs and confining vacua up to an additive anomaly. In the five-dimensional case described above we see that there is also a modular anomaly in the matrix model result but now of a multiplicative form. Interestingly under a general modular transformation

$$\tau \to \frac{a\tau + b}{c\tau + d} \tag{2.27}$$

$i\beta m$ transformations like a point on the torus with complex structure τ:

$$i\beta m \to \frac{i\beta m}{c\tau + d} . \tag{2.28}$$

3. The elliptic superpotential

Another method that has been used to calculate the exact values of the superpotential in all the massive vacua of the four-dimensional $\mathcal{N} = 1^*$ theory is to compactify it on a circle of finite radius [11, 12]. The effective superpotential is then a function of the dual photons and Wilson lines of the abelian subgroup $\mathrm{U}(1)^{N-1} \subset \mathrm{SU}(N)$. These comprise $N-1$ complex scalar fields X_a, $a = 1, \ldots, N$ (with $\sum_{a=1}^N X_a = 0$) which naturally live on a torus of complex structure τ because of the periodicity of each dual photon and Wilson line. The superpotential describing the $\mathcal{N} = 1^*$ deformation is therefore constrained to be an elliptic function of the complex scalars X_a. It turns out that this superpotential is identified as the basic hamiltonian of the elliptic Calogero-Moser integrable system where the X_a are position coordinates. The resulting superpotential, as originally found in [11] can be

$$W_{\text{elliptic}} = \mu \left(\sum_a \frac{1}{2} P_a^2 - m^2 \sum_{a \neq b} \wp(X_a - X_b) \right) , \tag{3.1}$$

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where $\wp(z)$ is the Weierstrass function defined on a torus with periods $2\pi i$ and $2\pi i\tau$. Notice that the momenta can trivially be integrated out. What is particularly useful about this superpotential is that it is independent of the compactification radius, and, therefore, yields results that are valid in the four-dimensional limit. In addition unlike the matrix model approach it encodes all the vacua of the $\mathcal{N}=1^*$ theory, both massive and massless, in a single superpotential.

The question is how this elliptic superpotential is generalized in the five-dimensional theory compactified on a circle. In order to motivate the answer we need to recall in more detail why the elliptic Calogero-Moser integrable system appears in the basic $\mathcal{N}=1^*$ case. The reason is that, if we think of the $\mathcal{N}=1^*$ theory in terms of a deformation by the mass term μ of the $\mathcal{N}=2^*$ theory, then the Coulomb branch of this latter theory is described by the moduli space of a Seiberg-Witten curve. This curve is precisely the spectral curve of the elliptic Calogero-Moser system [18]. Now for the five-dimensional theory, Nekrasov has made a conjecture for the Seiberg-Witten curve: it is precisely the spectral curve of a one parameter deformation of the elliptic Calogero-Moser system known as the elliptic Ruijsenaars-Schneider system. This is sometimes known as the relativistic elliptic Calogero-Moser system.

Following the approach of [11, 12] it is natural to conjecture that the elliptic superpotential of the five-dimensional theory, for the simplest deformation $\mu\cosh\beta\Phi$, is the basic hamiltonian of the Ruijsenaars-Schneider system. Based on this, and with reference to [13], we expect

$$W_{\text{elliptic}} = C\mu\sum_a \cosh(\beta P_a)\prod_{b\neq a}\sqrt{\wp(i\beta m)-\wp(X_a-X_b)}\,, \tag{3.2}$$

for some constant C. It is easy to see that the critical points which describe the confining vacuum of the $\mathcal{N}=1^*$ case

$$P_a = 0\,, \qquad X_a = \frac{2\pi i a\tau}{N}\,, \qquad a=1,\ldots,N\,, \tag{3.3}$$

are still critical points of the superpotential (3.2). In fact this is true for all the massive vacua described in [11].

We can now compare the result of the quantum mechanical matrix model (2.24) with the result from the elliptic superpotential:

$$W_{\text{elliptic}} = NC\mu\prod_{a=1}^{N-1}\sqrt{\wp(i\beta m)-\wp\left(\frac{2\pi i a\tau}{N}\right)}\,. \tag{3.4}$$

Amazingly the results agree by virtue of the elliptic function identity (A.8) proved in the appendix

$$\frac{\theta_1(\frac{\beta}{2}|\frac{\tau}{N})}{\theta_1'(0|\frac{\tau}{N})} = (2i)^{N-1}\left(\frac{\theta_1(\frac{\beta}{2}|\tau)}{\theta_1'(0|\tau)}\right)^N\prod_{a=1}^{N-1}\sqrt{\wp(i\beta m|\tau)-\wp\left(\frac{2\pi i a\tau}{N}|\tau\right)}\,, \tag{3.5}$$

where we have emphasized the complex structure associated to the elliptic functions on the right-hand side.

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A. Some properties of elliptic functions

We use the (quasi-)elliptic functions $\wp(u)$, $\zeta(u)$, $\theta_1(\frac{u}{2i}|\tau)$ associated to a torus of periods $2\omega_1 = 2\pi i$ and $2\omega_2 = 2\pi i\tau$ as defined in [17]. An important equation relating them is

$$\zeta(u) - \frac{\zeta(\omega_1)}{\omega_1}u = \frac{1}{2i}\frac{\theta_1'(\frac{u}{2i}|\tau)}{\theta_1(\frac{u}{2i}|\tau)}\,, \tag{A.1}$$

We also use the heat equation

$$\frac{\partial^2\theta_i(x|\tau)}{\partial x^2} + \frac{4}{\pi i}\frac{\partial\theta_i(x|\tau)}{\partial\tau} = 0 \tag{A.2}$$

and the relations

$$\zeta(\omega_1)\omega_1 = \frac{\pi^2}{12}\cdot E_2(\tau) = -\frac{\pi^2}{12}\cdot\frac{\theta_1'''(0|\tau)}{\theta_1'(0|\tau)}\,. \tag{A.3}$$

In the remainder of the appendix, we establish a crucial identity. We start with the following identity established in [7] (using (A.1), (A.3) and other standard properties of elliptic functions)

$$\frac{\theta_1'(\frac{u}{2i}|\frac{\tau}{N})}{\theta_1(\frac{u}{2i}|\frac{\tau}{N})} - \frac{\theta_1'(\frac{u}{2i}|\tau)}{\theta_1(\frac{u}{2i}|\tau)} + \frac{iu(N-1)}{6}E_2(\tau) = i\sum_{a=1}^{N-1}\left(\zeta\left(\frac{2\pi ai\tau}{N}+u\right) - \zeta\left(\frac{2\pi ai\tau}{N}-u\right)\right). \tag{A.4}$$

We now integrating this expression, using the relation (A.1) on the right-hand side, to arrive at

$$\frac{\theta_1(\frac{u}{2i}|\frac{\tau}{N})}{\theta_1(\frac{u}{2i}|\tau)} = \prod_{a=1}^{N-1}\sqrt{\theta_1\left(\frac{\pi a\tau}{N}-\frac{u}{2i}|\tau\right)\theta_1\left(\frac{\pi a\tau}{N}+\frac{u}{2i}|\tau\right)}\,. \tag{A.5}$$

Further, on the right-hand side, we can employ the relation

$$\frac{\theta_1(\frac{u-v}{2i}|\tau)\theta_1(\frac{u+v}{2i}|\tau)}{\theta_1(\frac{u}{2i}|\tau)^2\theta_1(\frac{v}{2i}|\tau)^2} = \wp(v) - \wp(u) \tag{A.6}$$

to derive the identity

$$\theta_1\left(\frac{u}{2i}|\frac{\tau}{N}\right) = \theta_1\left(\frac{u}{2i}|\tau\right)^N\prod_{a=1}^{N-1}\theta_1\left(\frac{\pi a\tau}{N}|\tau\right)\sqrt{\wp(u)-\wp\left(\frac{2\pi ia\tau}{N}\right)}\,. \tag{A.7}$$

From the ratio of this, and its derivative evaluated at $u=0$, we have our crucial identity

$$\frac{\theta_1(\frac{u}{2i}|\frac{\tau}{N})}{\theta_1'(0|\frac{\tau}{N})} = (2i)^{N-1}\left(\frac{\theta_1(\frac{u}{2i}|\tau)}{\theta_1'(0|\tau)}\right)^N\prod_{a=1}^{N-1}\sqrt{\wp(u)-\wp\left(\frac{2\pi ia\tau}{N}\right)}\,. \tag{A.8}$$

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